INTRODUCTION

Tessellation - what does it all mean? The word comes from the Latin, 'Tessara', meaning a 'tile' in a mosiac pattern. This project is mainly concerned with tessellations on flat surfaces, although other possibilities can be investigated. Tessellations do not have to be on their own, many different shapes can be used to produce some quite intricate designs. Here are a few of the possibilities :-

1) Regular Shapes (equal sides and equal angles)
   a) Single
   b) Mixed

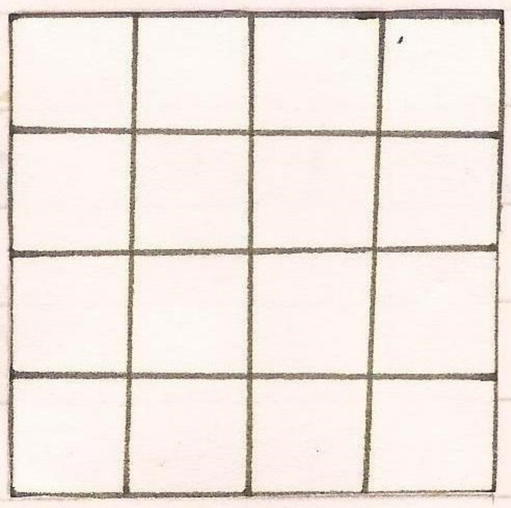
2) Irregular Shapes (unequal sides and unequal angles)
   a) Straight edges
   b) Curved edges

3) Designs - M.C. Escher

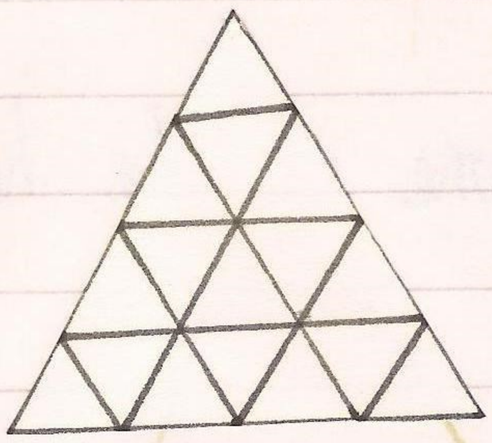
INVESTIGATIONS

Single Regular Shapes

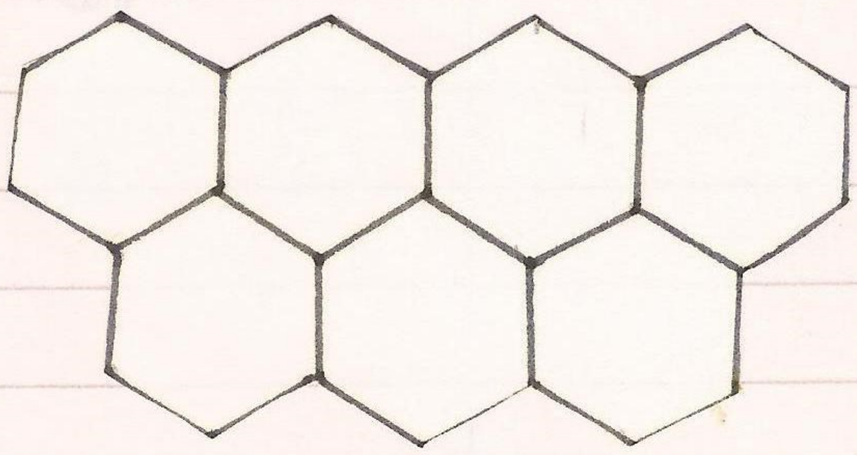
By experimentation, it is easy to discover that these polygons tessallate:-



squares

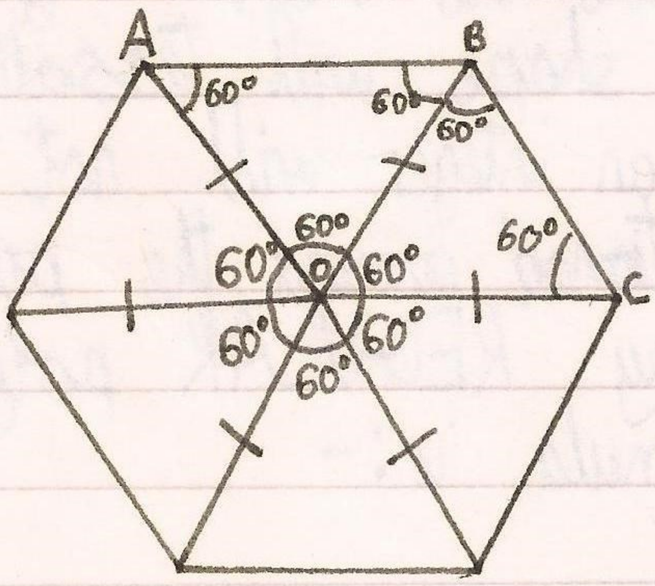


triangles



hexagons

But why? Are these the only ones, are there more? To answer these questions you need to investigate the angles. Take, for example, this hexagon:-



Divide the hexagon into triangles. The angles around the centre will each be 60° (360° ÷ 6) and will naturally total 360°. As they are triangles it is possible to calculcate the other angles. Line AO and BO will both be equal, so ∠OAB and ∠OBA will also both be equal

$\therefore \angle OAB + \angle OBA + 60° = 180°$

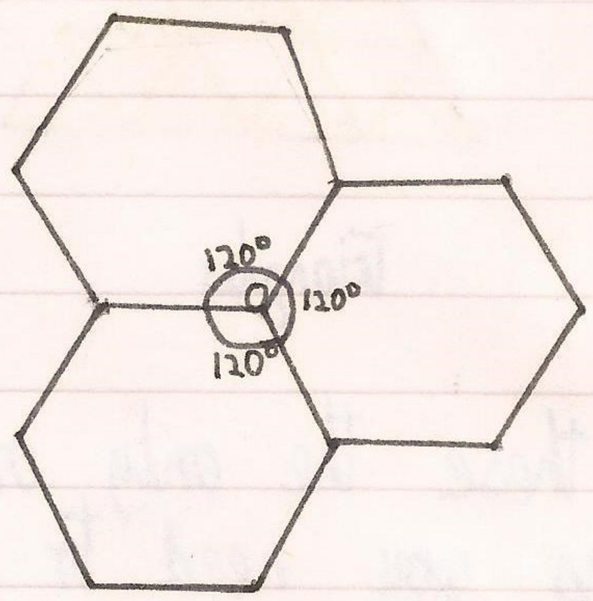
$\therefore \angle OAB + \angle OBA = 120°$

$\therefore \angle OAB = \frac{120°}{2} = 60° \quad \angle OBA = \frac{120°}{2} = 60°$

This also means that the angles in the other 7 triangles will be 60°.

So, if $\angle OBA = 60°$
and $\angle OBC = 60°$
then $\angle ABC = 120$

$\angle ABC$ is one of the hexagon's internal angles



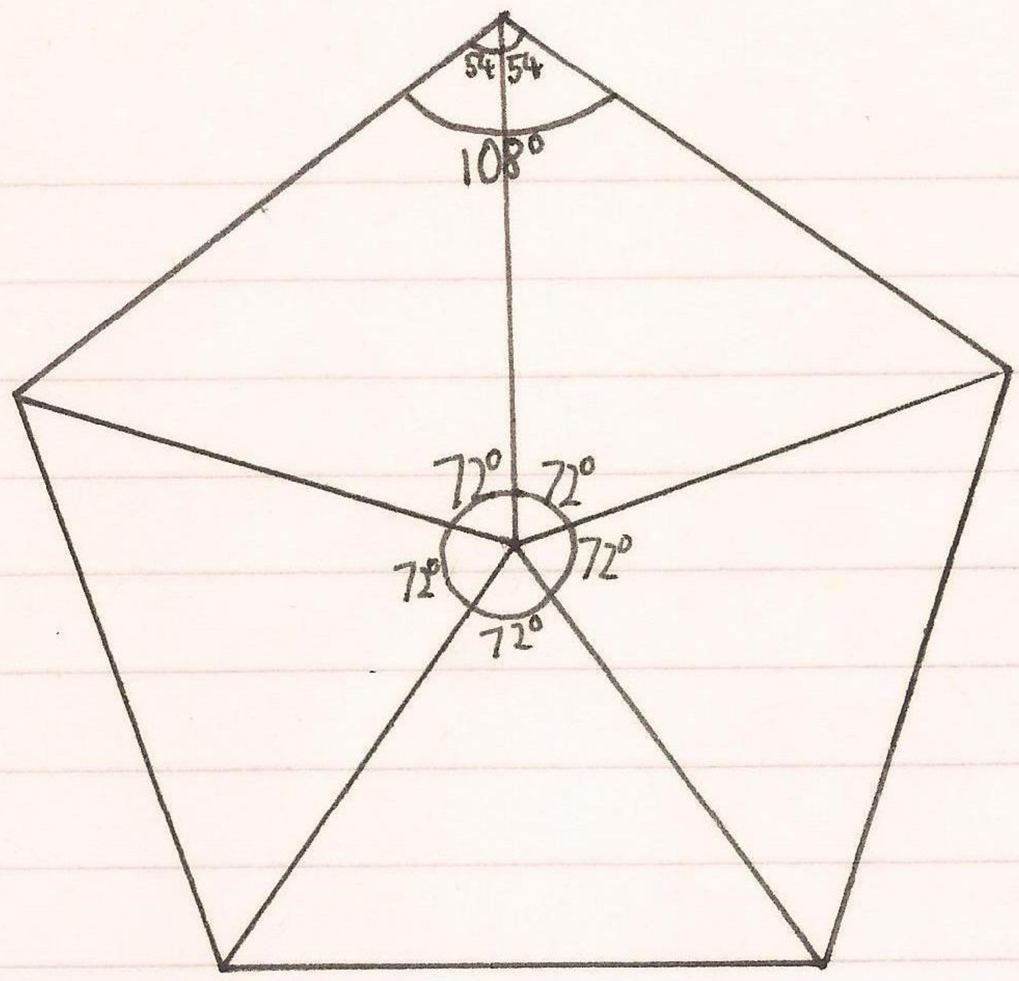
Now if you divide the internal angle into 360° if your answer is an integer the shape will tessallate. This means that if the answer was an integer will not leave a gap when the polygon is positioned around the point O. This procedure can be done with any REGULAR polygon. In algebraic notation the formula is:-

Key:- S = Sides

If $\dfrac{360}{180-(360 \div S)}$ = integer: shape tessalates

This can be modified to calculate the internal in a slightly different way:-

If $\dfrac{360}{(S-2) \times 180 \div S}$ = integer: shape tessallates



A Pentagon

Obviously, pentagons can't tessallate but can my rule prove this? That is what I am about to find out.

So, the angles around the centre are each 72°. (360/5)
∴ Each triangle angle is 54° ($\frac{180-72}{2}$)
∴ Internal angle is 108° (54×2)
∴ Shape does NOT tessallate ∵ $\frac{360}{108} = 3.\dot{3}$

MIXED REGULAR SHAPES

It is also possible to use 2 or more regular shapes in conjunction with each other to form a new tessellating pattern

e.g.

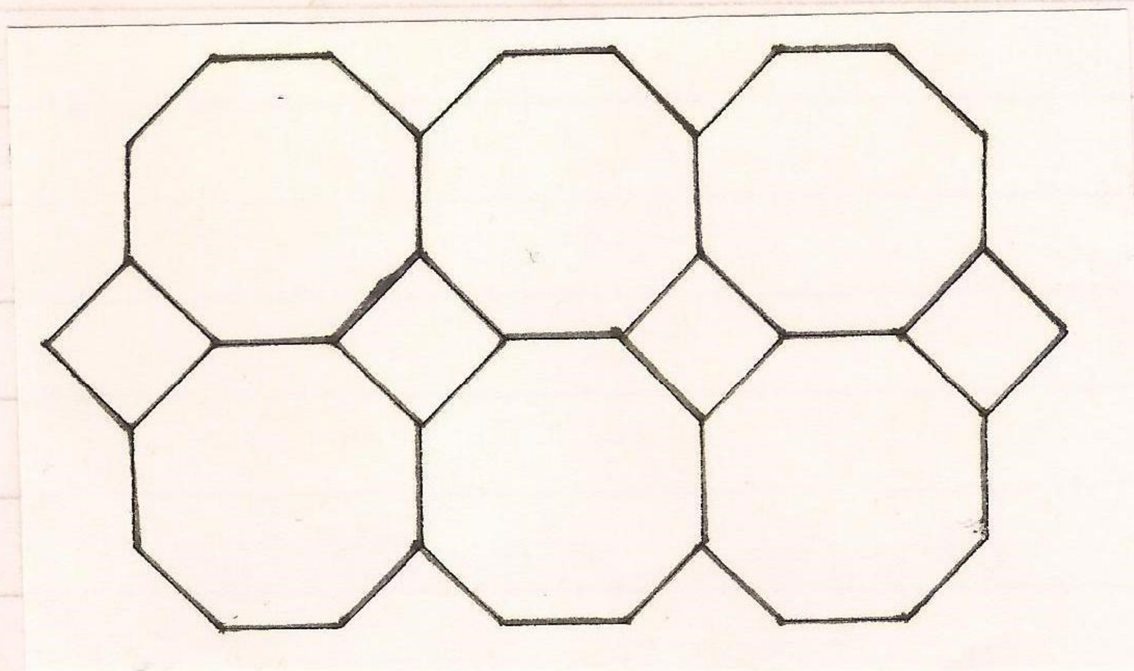


Figure 1

Octogons & squares

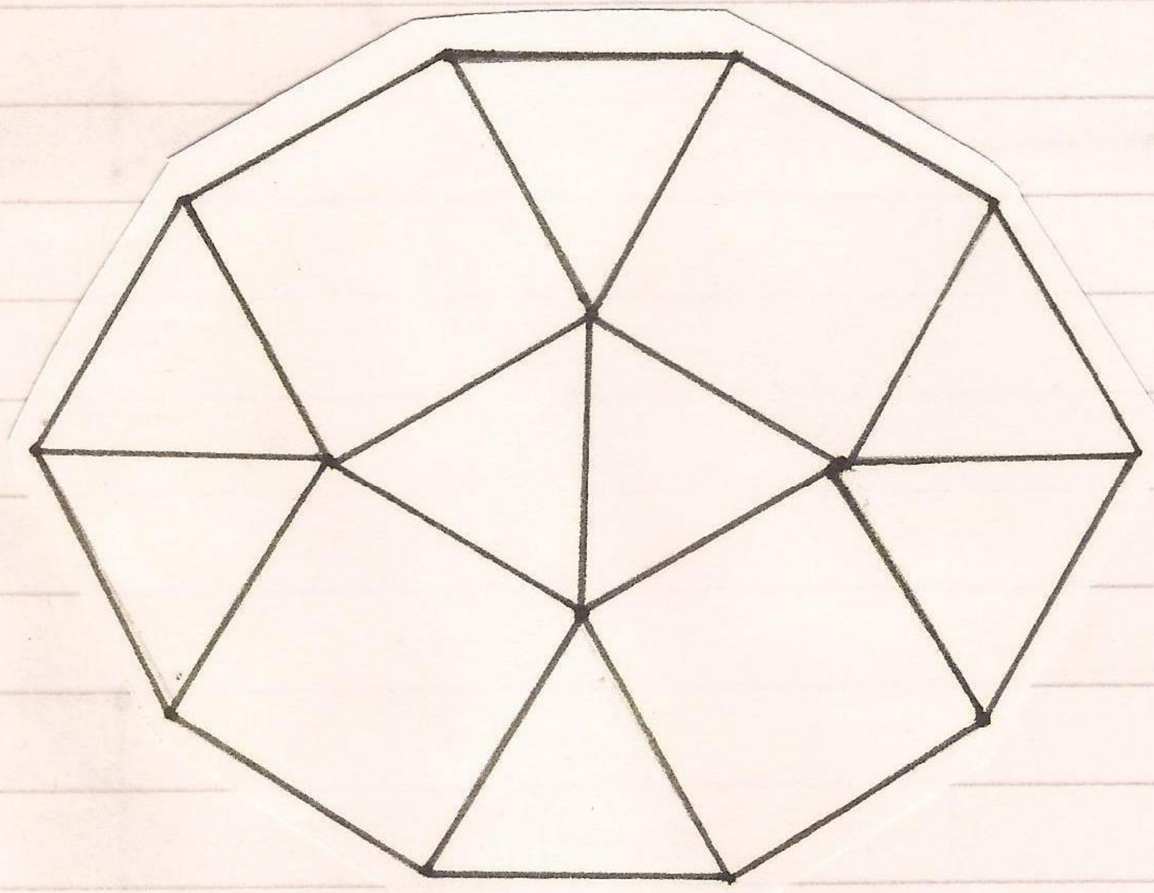


Figure 2

Squares and triangles

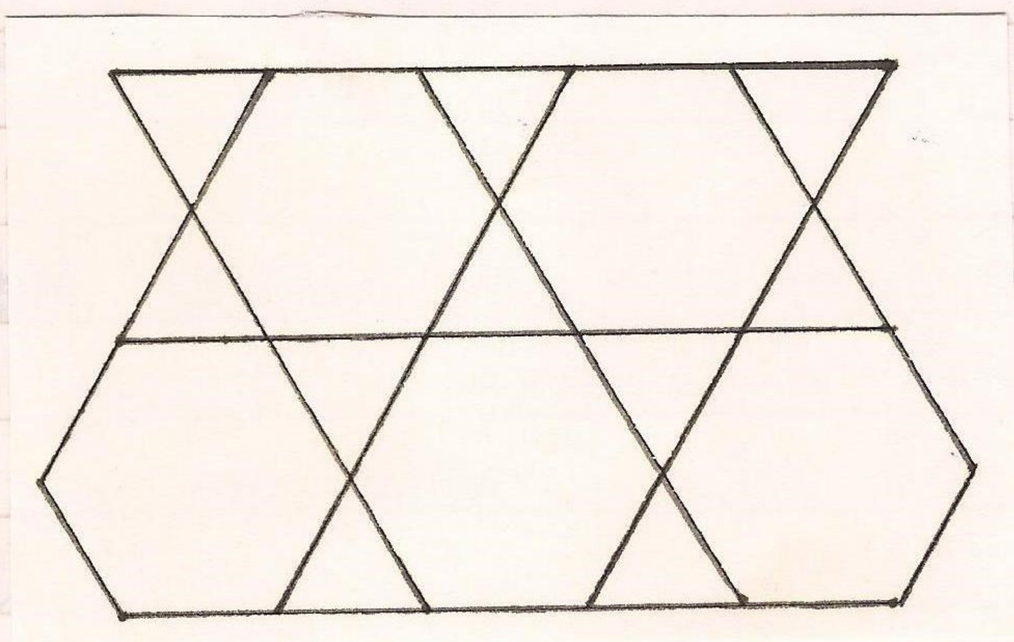


Figure 3

Hexagons and triangles

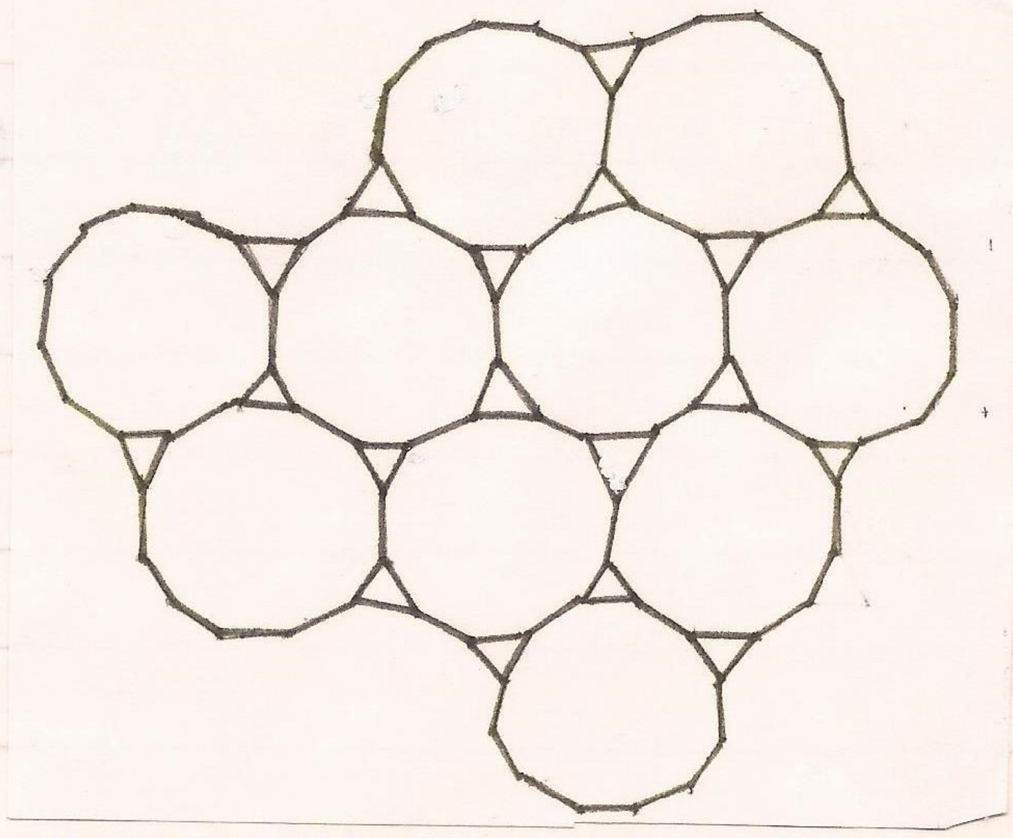
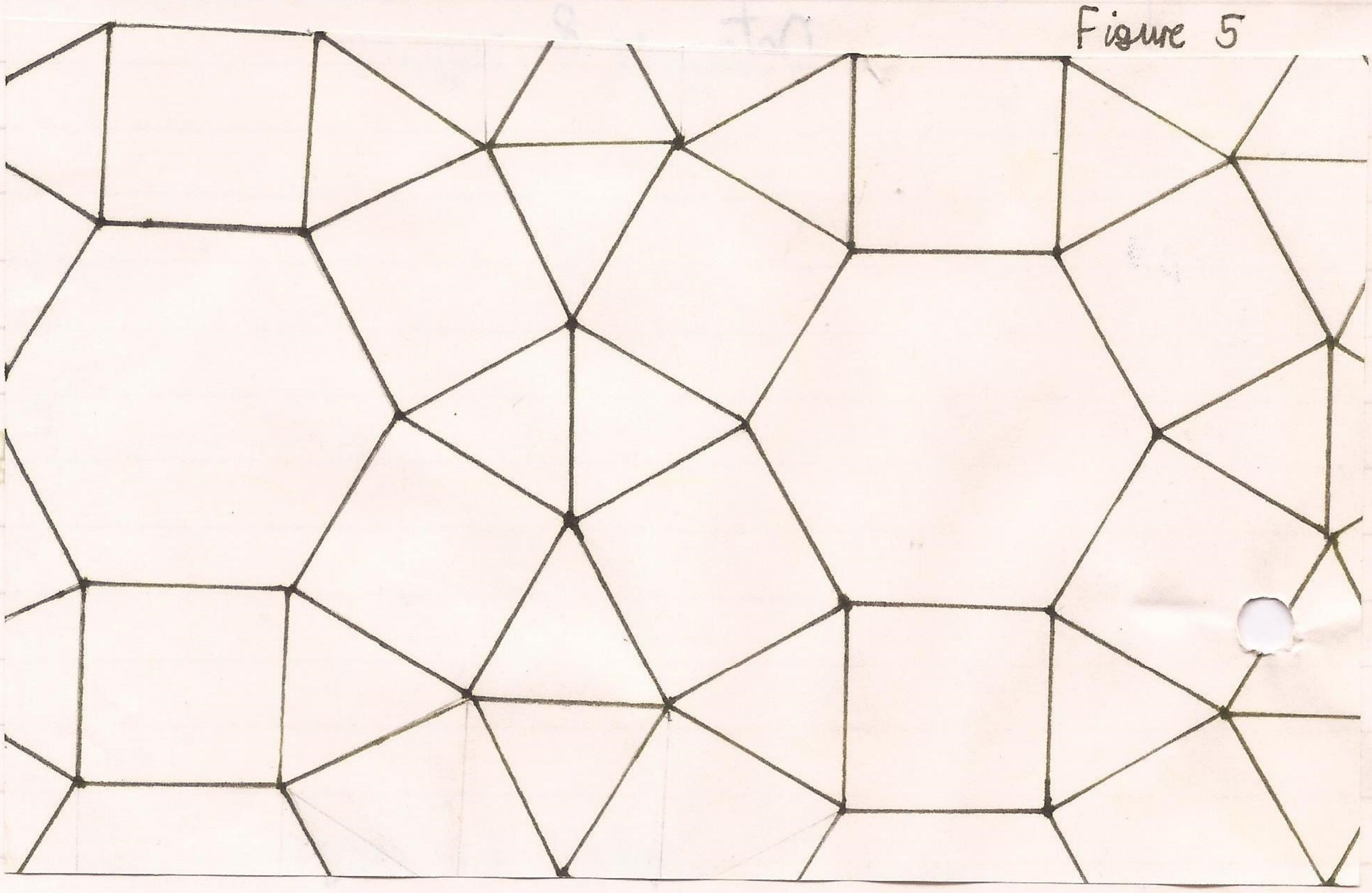


Figure 4

Dodecagons and triangles

Figure 5



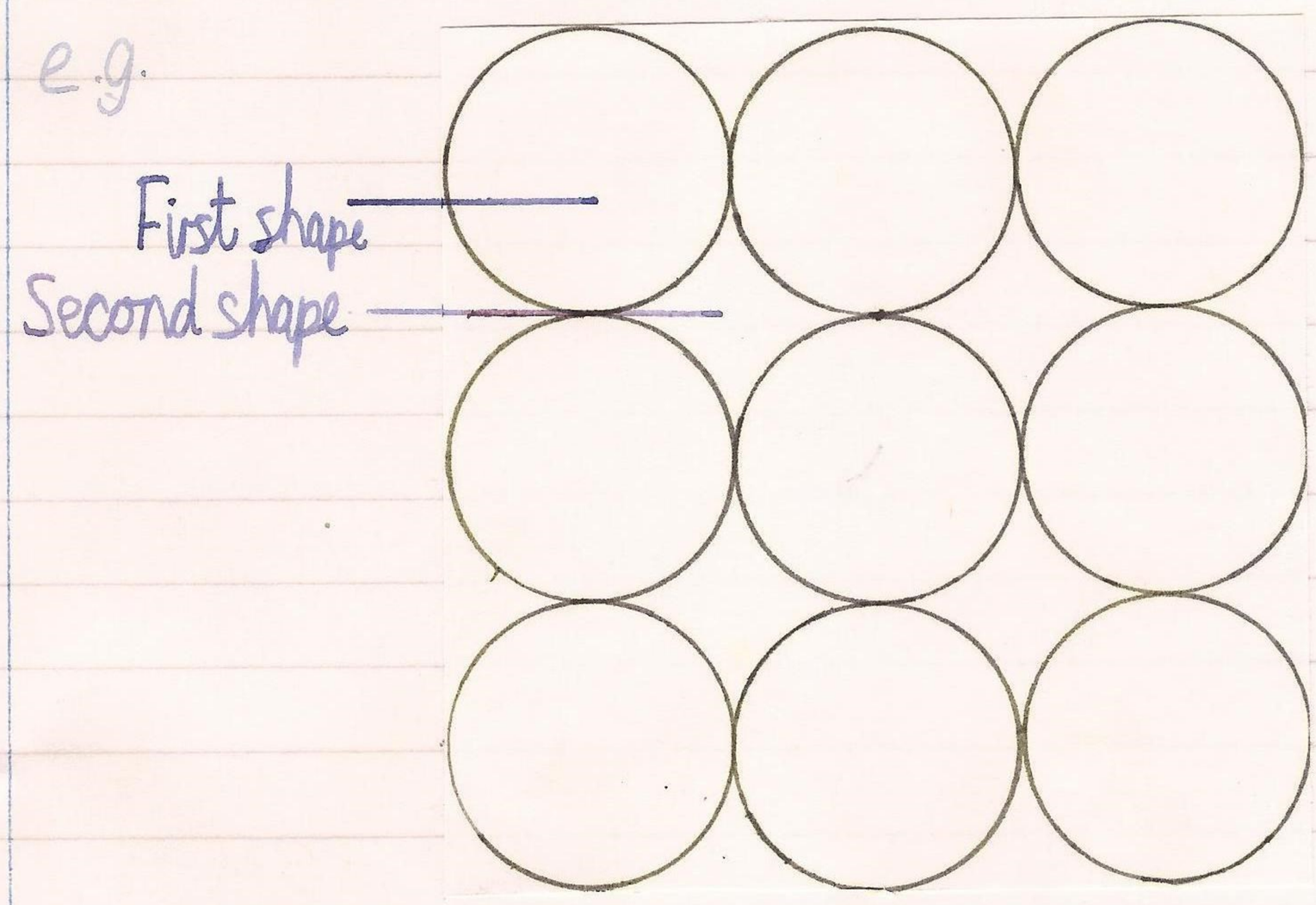
Hexagons, squares and triangles.

In this tessellation, the squares and triangles around each hexagon create dodecagons and the gaps between them are filled in with extra triangles, and this is the tessellation in Figure 4. It is interesting to see that a tessellation can be made with squares, triangles and hexagons as these are the only regular tessellating polygons.

CREATING MIXED TESSELLATIONS

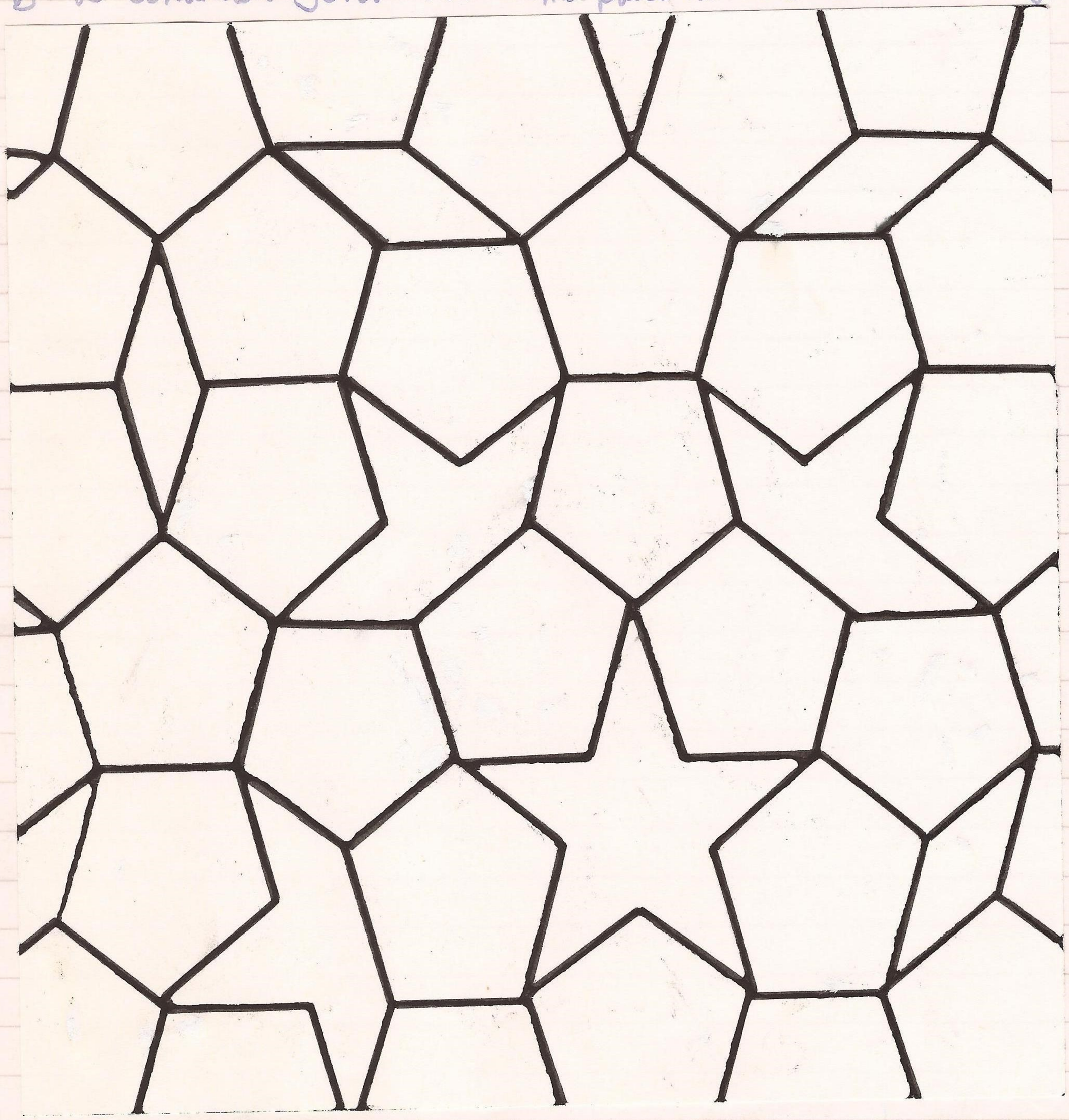
You can always make a tessellation with any polygon and the polygon corresponding to the gaps left between the first set of polygons. When the first set of polygons is put down, each one must be at exactly the same angle and the same level not going up or down (except when a new line is started) or the second polygon will not always be the same. If the first polygon already tessalates on its own, the angle which it faces must be changed to make it leave a gap.

e.g.



PENROSE TILING

This was created by Dr. Penrose of Oxford University. He was trying to figure out a way of using regular pentagons in a tessalation pattern. He uses stars, hats, diamonds and rhombuses with the pentagons in the pattern. It is intriguing as it never actually repeats itself, even if it continued forever. In some place he does use a mirror image though.



CREATING A TESSELATING SHAPE

First of all you need to get any shape that already tesselates, then move a piece any size, any shape to the corresponding place on the other side of the shape. This process can be done again and again to the shape and then no-one could guess what shape you started with. Here is an example

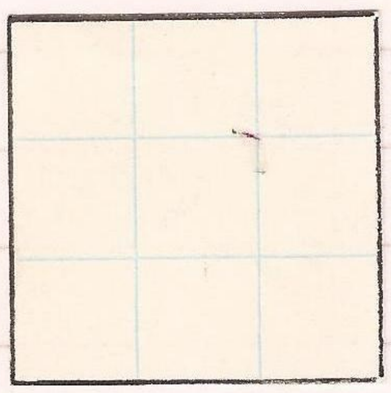


FIGURE 1

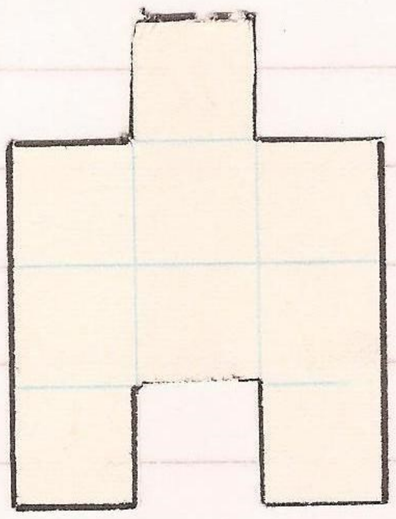


FIGURE 2

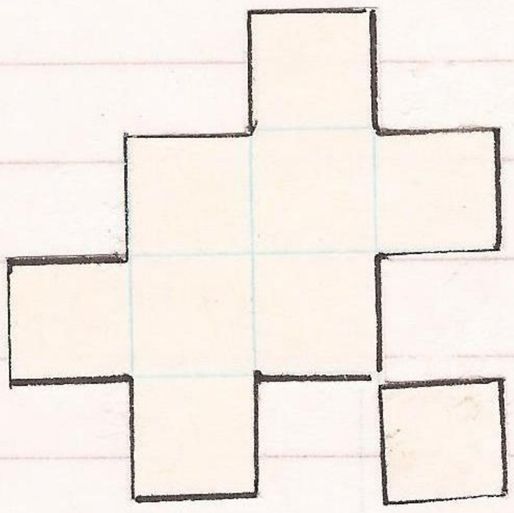


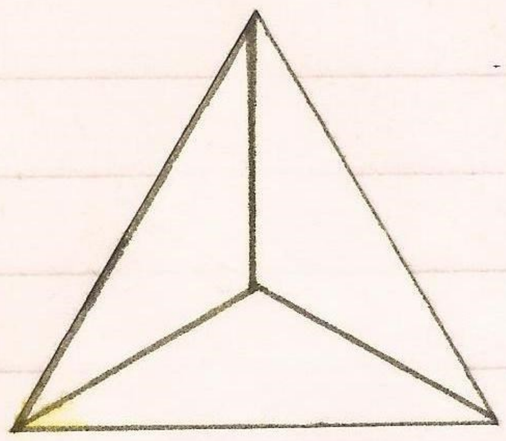
FIGURE 3!

The way this method works is the original shape tesselates as usual, BUT the added on bit will fit into the gap. So if you know how the original shape tesselates, you do not need to work out how the new shape tesselates.

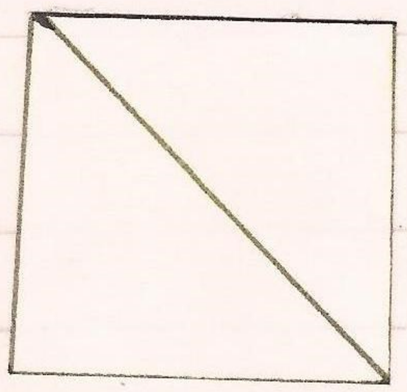
IRREGULAR SHAPES

Tessellating regular shapes can be subdivided into irregular shapes to form a new tessellation.

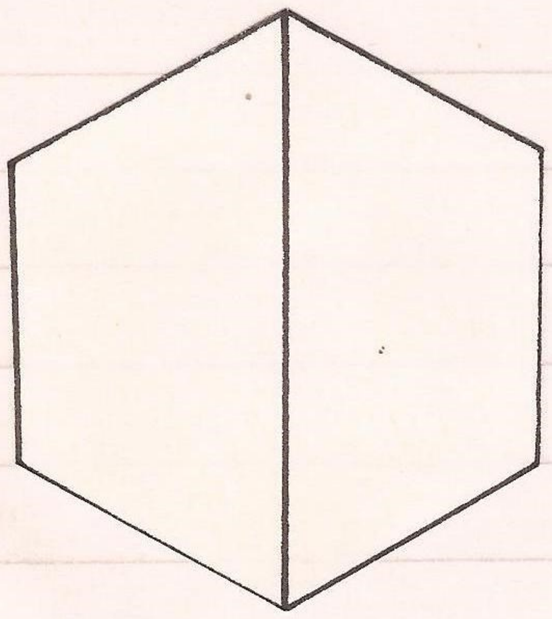
e.g.



Equilateral triangle
into
3 Isosceles triangles



Square
into
2 right angle triangles

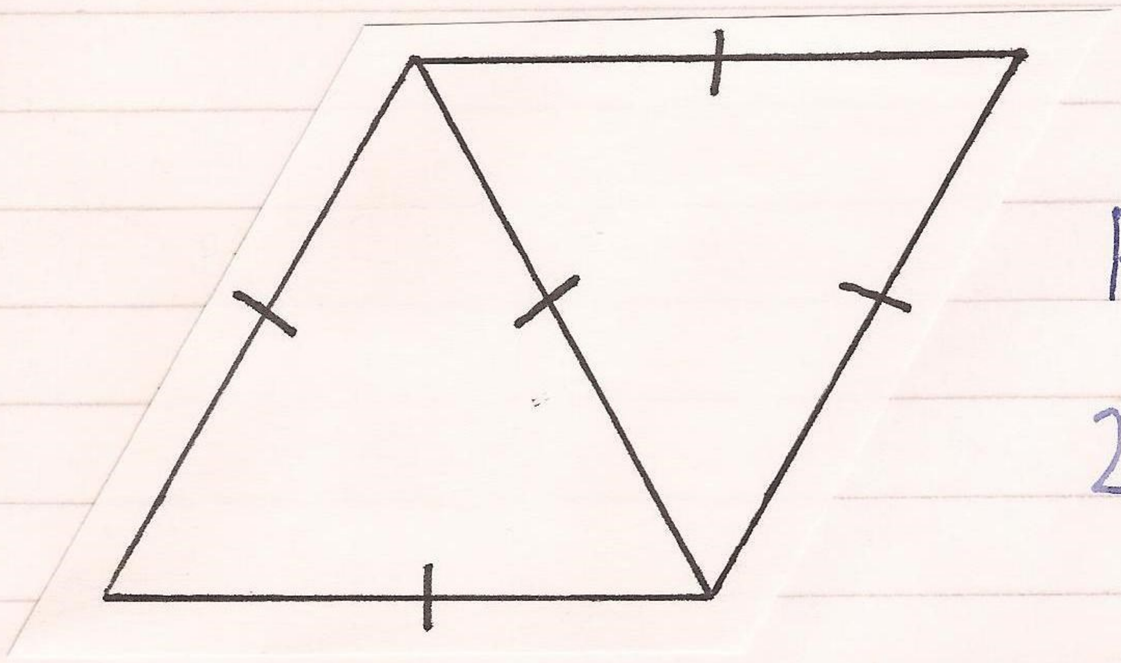


Regular hexagon
into
2 trapezia

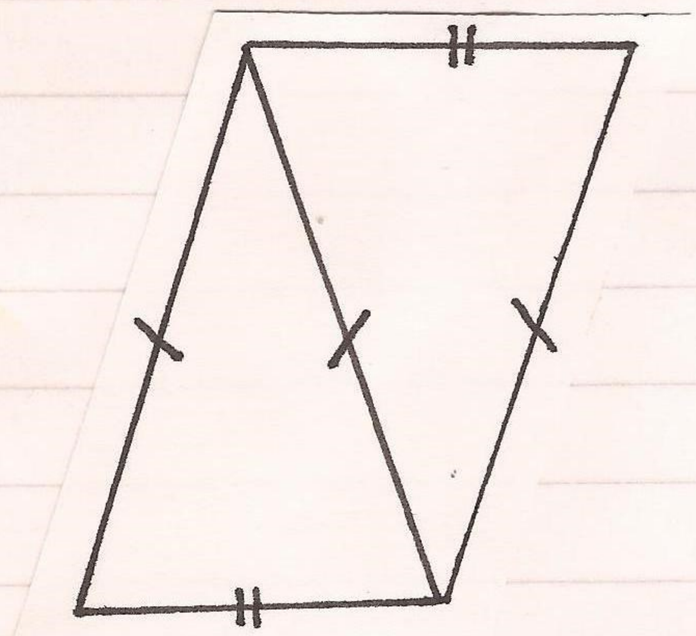
Of course, these polygons can be split into different shapes instead. For example, a hexagon can be split into 2 isosceles triangles (one at the top and one at the bottom) and a rectangle. There are many different possibilities. In fact, it can be an irregular shape that you split up, but it has got to be one that tessellates.

CAN ALL TRIANGLES TESSELLATE?

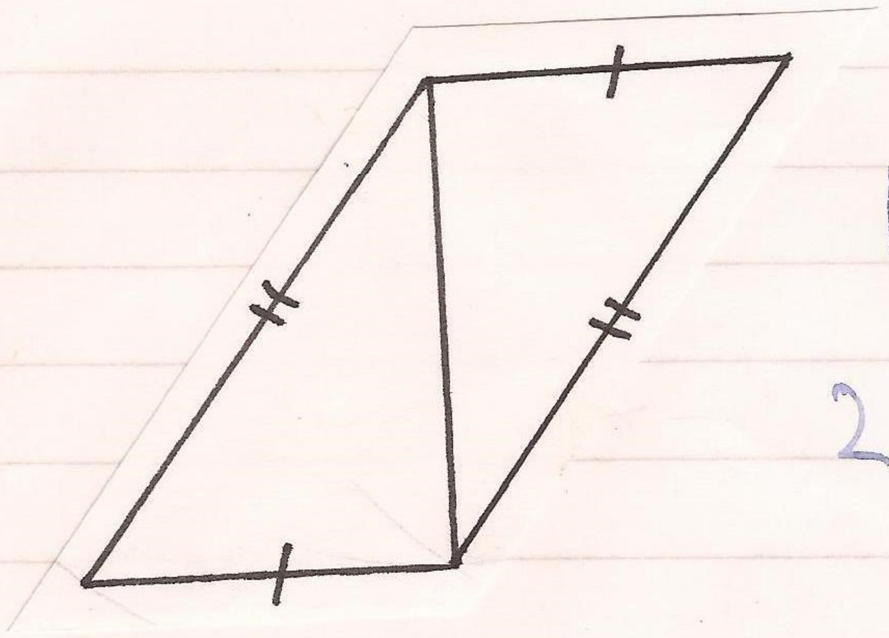
From the last section you know how tessellating shapes can be split up. Well all squares, rhombuses, rectangles and parallelograms can be split up to produce triangles - equilateral, isosceles and even scalene. Here is how:-



Rhombus
into
2 Equilateral triangles



Parallelogram
into
2 Isosceles triangles



Parallelogram
into
2 Scalene triangles

All this means is that if a rhombus can tessellate, then half a rhombus can also tessellate, and if a parallelogram can tessellate,

half a parallelogram can tessellate.

AMAZING, BUT TRUE!

Although I've prooved you can't tessellate regular pentagons, it IS possible to do it with an irregular one that closely resembles a regular pentagon, but with oprly one axis of symmetry. To do this, start with a tessellation of elongated hexagons (fig 1), and place it over another identical one at 90° (fig 2)

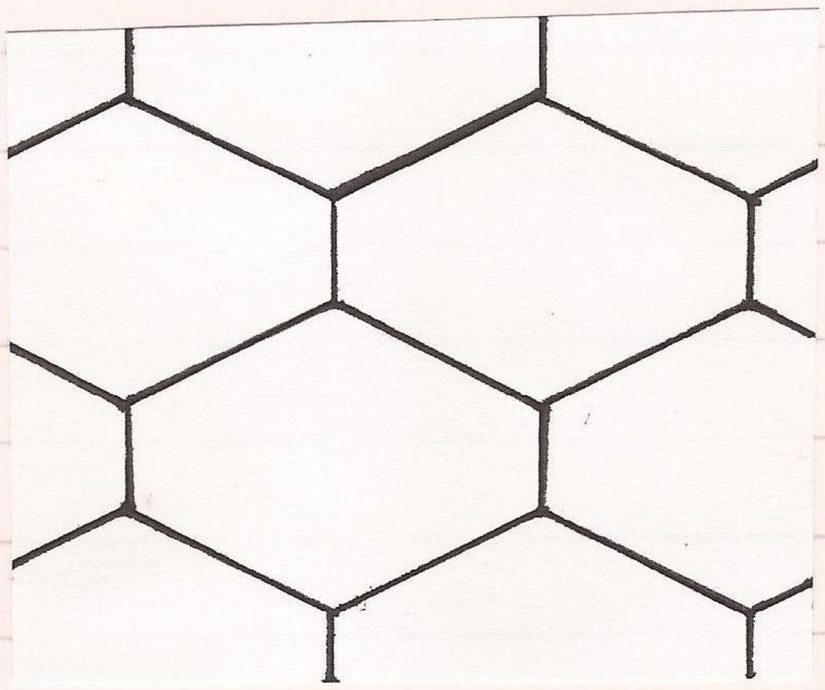


Figure 1.

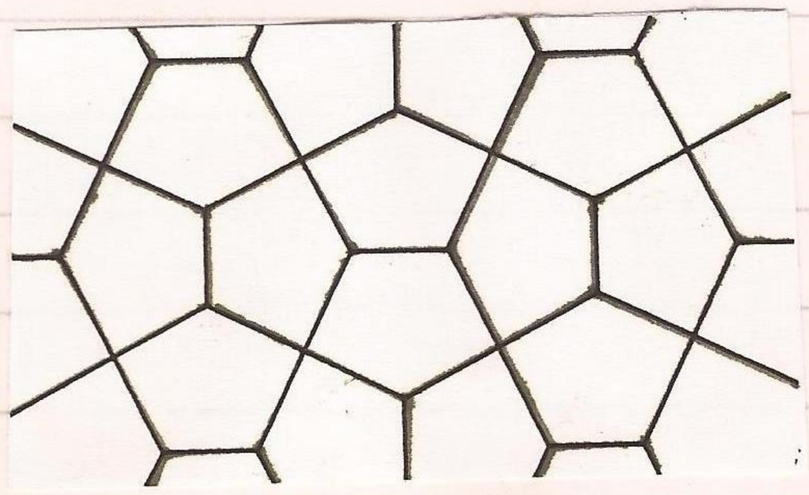
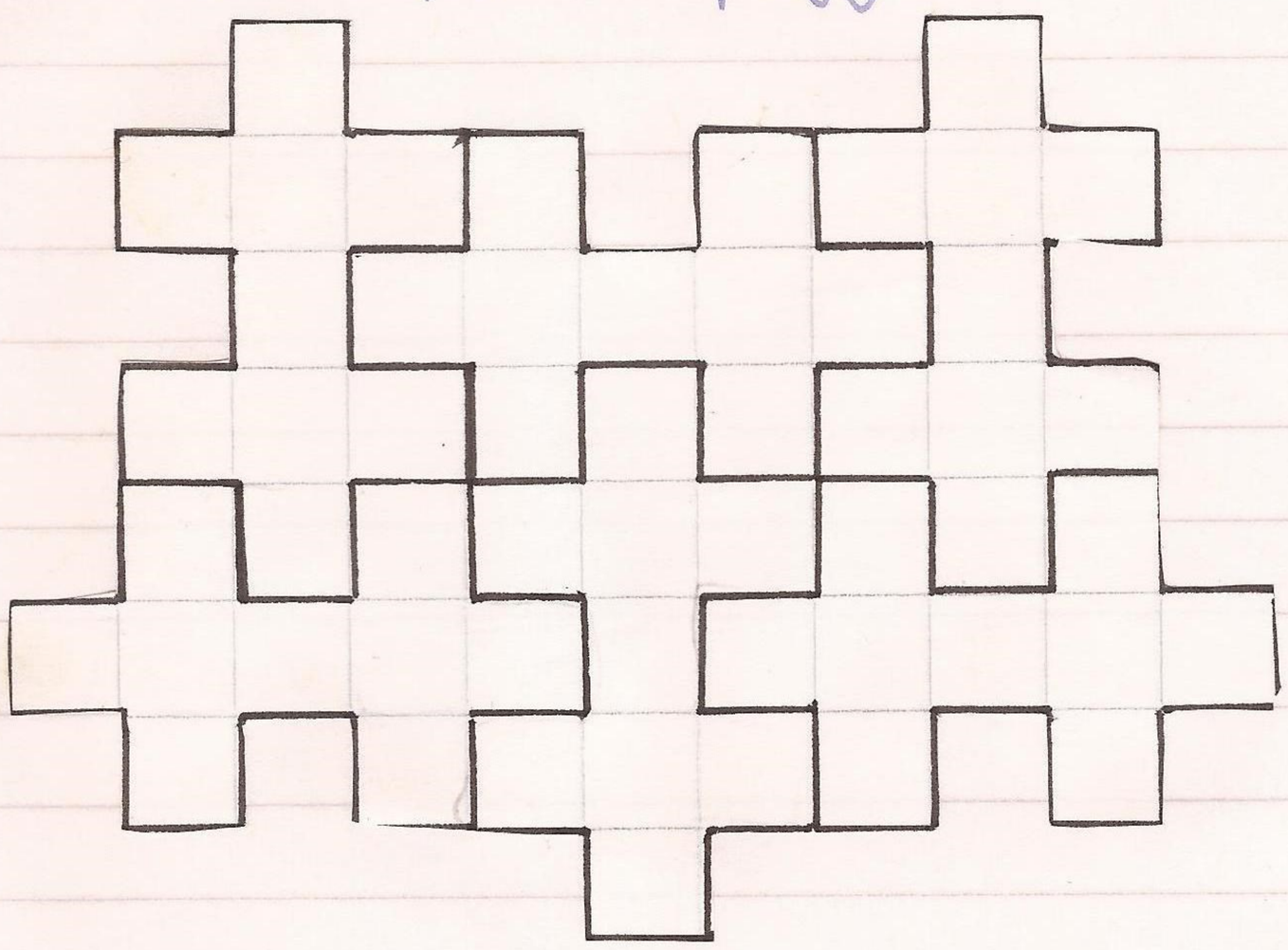


Figure 2.

RE-ENTRANT SHAPES

Re-entrant shapes are polygons with reflex angles.

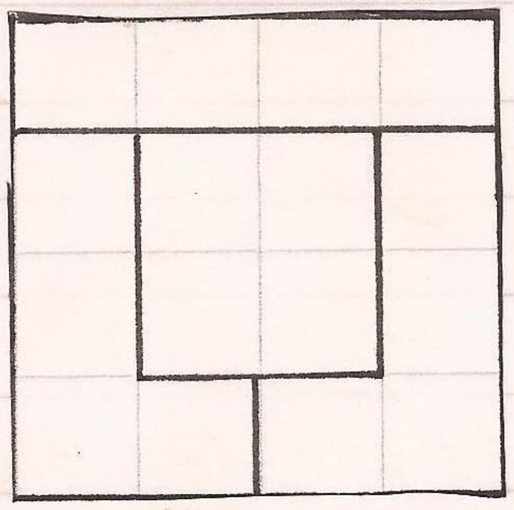
e.g.



This is an example of a very famous type of shape called polyominoes. i.e. A shape formed from squares joining together edge to edge.

They can either tessellate on their own (like above) or can be fitted together to make other tessellating shapes, such as squares or rectangles

e.g.

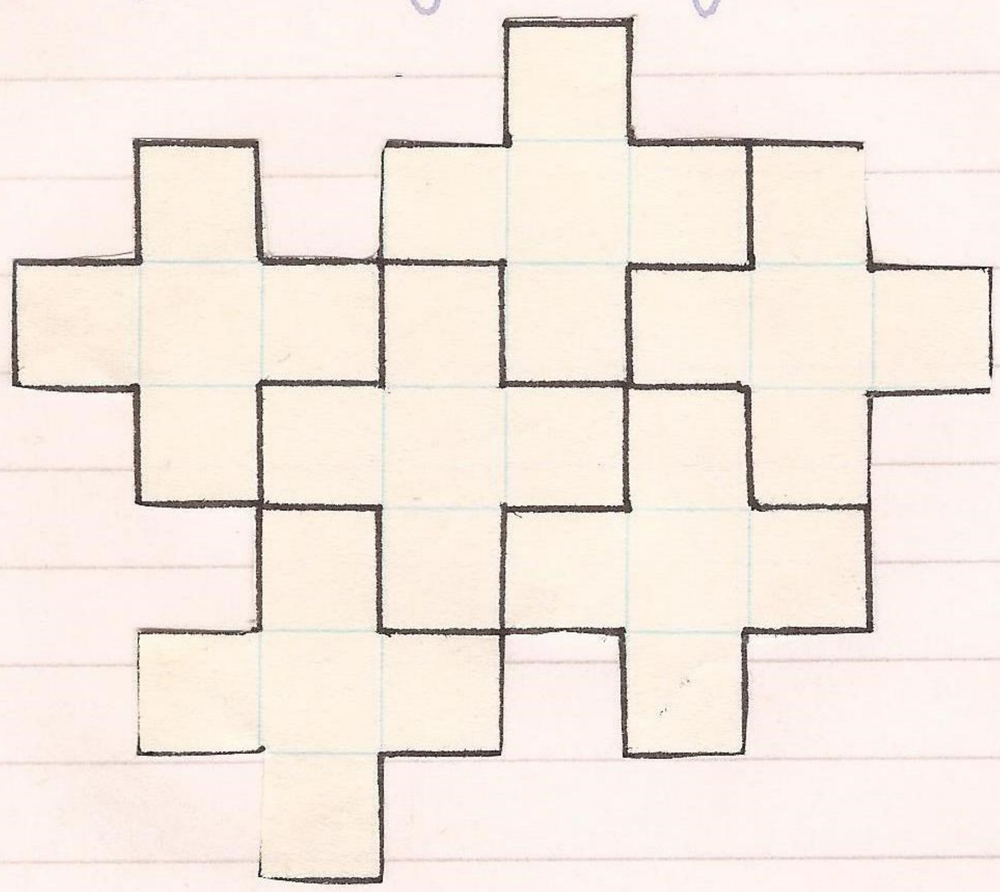


These are tetrominoes, from the Greek - 'tetra' meaning four.

TESSELLATION AREA

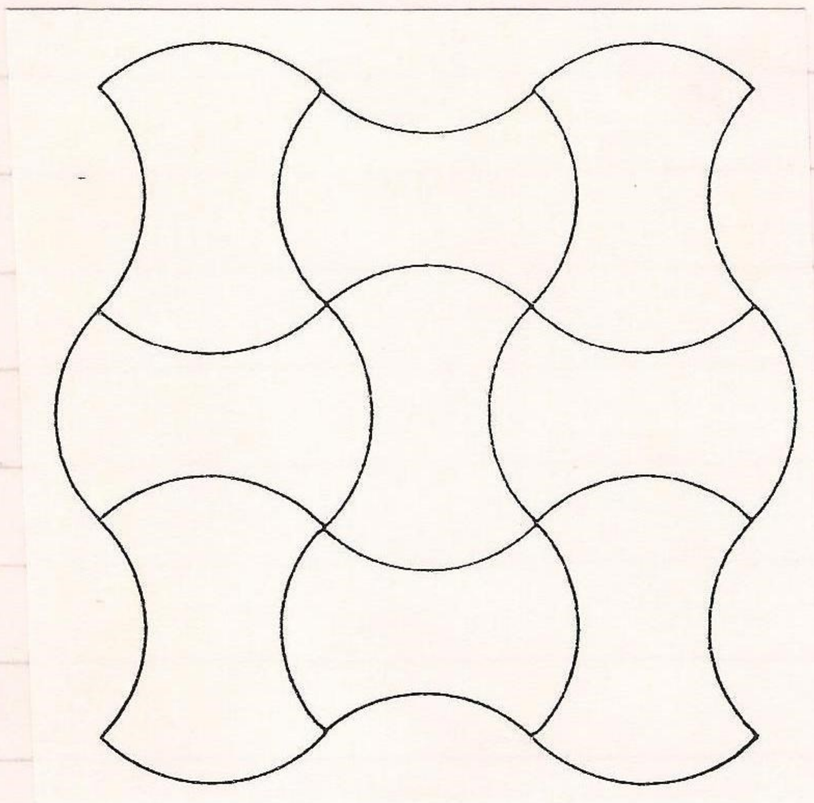
In principle, tessellations continue for an infinite area, but as this would be impossible to draw, in practice they will have irregular edges

e.g.



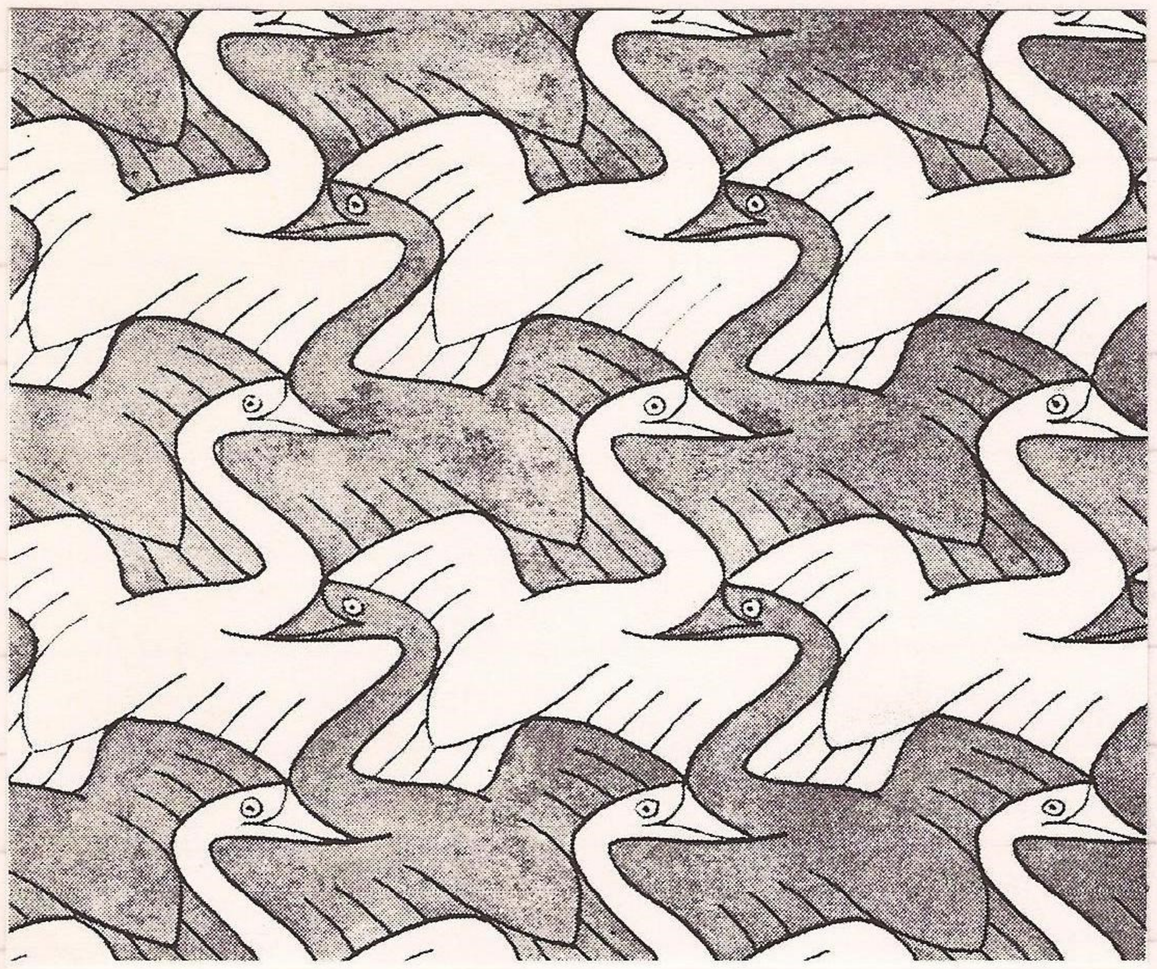
CURVED IRREGULAR SHAPES

All the examples so far have been straight edged. However, they can consist of curved sided



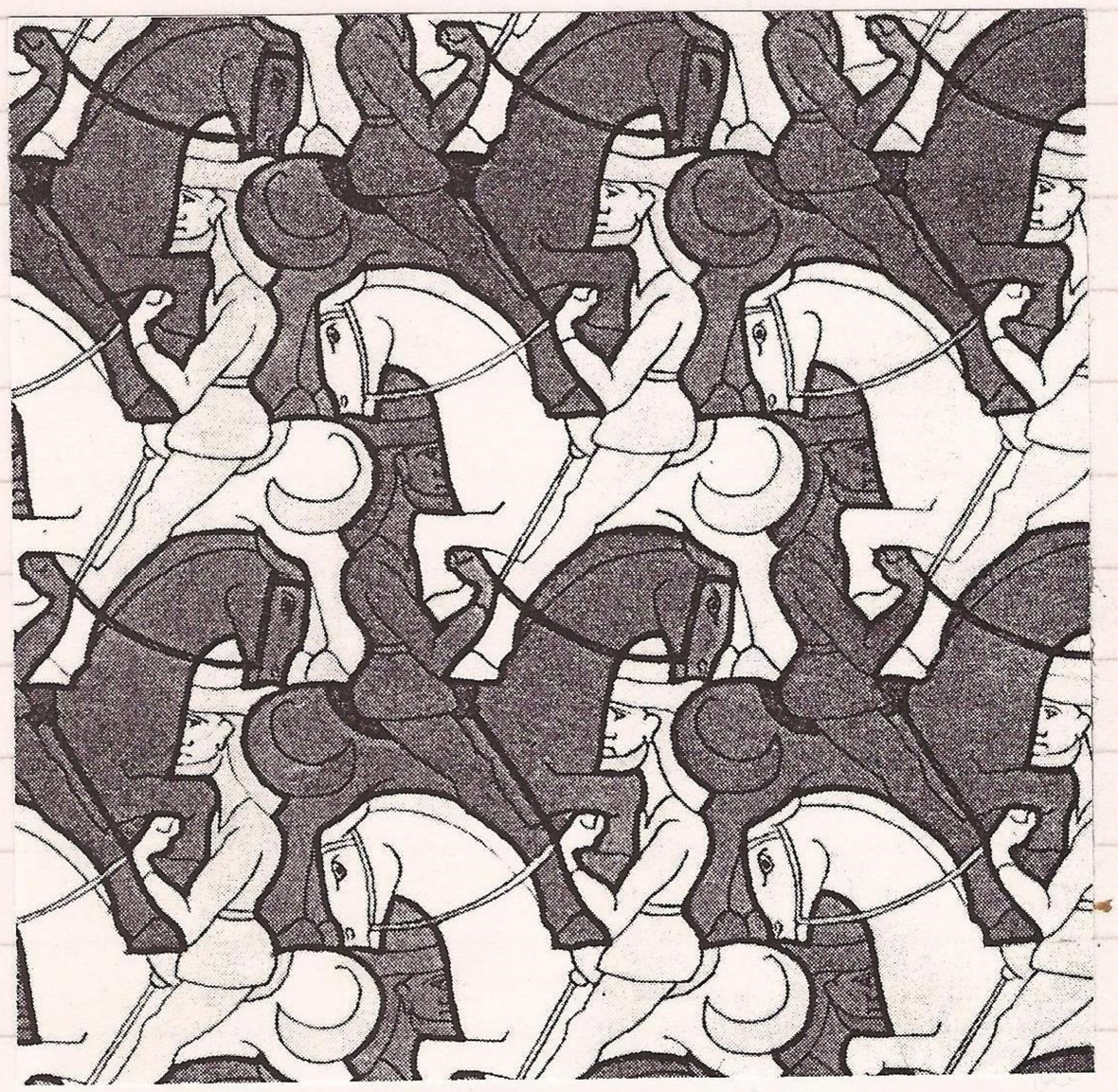
Here is one of the simpler examples. Look at the designs by M.C. Escher for something really complicated!!!

THE WORLD OF M.C. ESHER

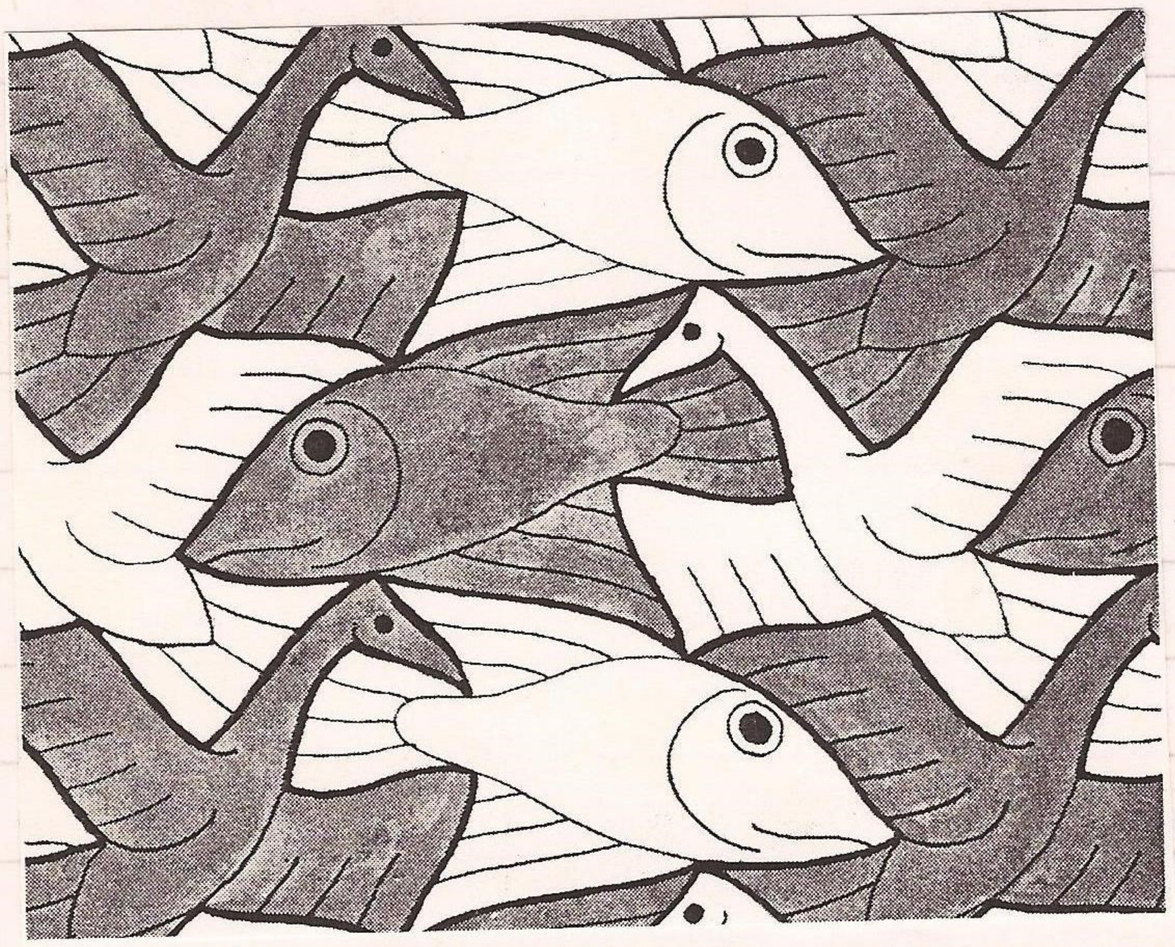


M.C. Esher was born in 1898 in Leeuwarden, Holland. During his life he lived in Switzerland and Belgium, then died aged 73. On the next few pages are some of his fractals. Remember, he had no computers to calculate them!

Swans



Horses



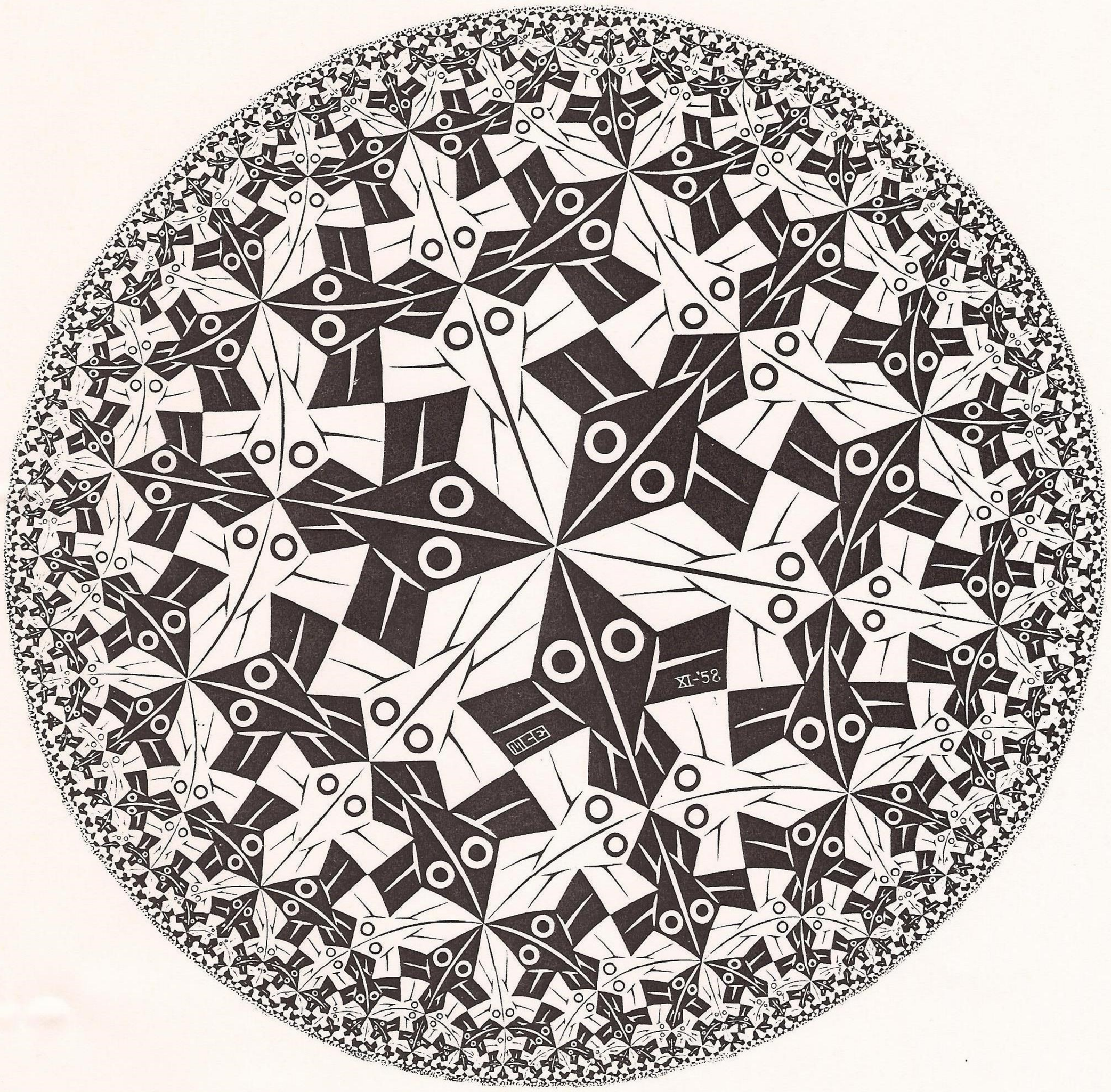
Fish and birds.

Angels and Devils

This design is a fractal. It gets smaller and smaller towards the edge. This may look circular, but is not. The detail is just as great at the edge.

25.
Kreislimit IV
Circle limit IV
Limite circulaire IV
Cirkellimiet IV
Límite circular IV

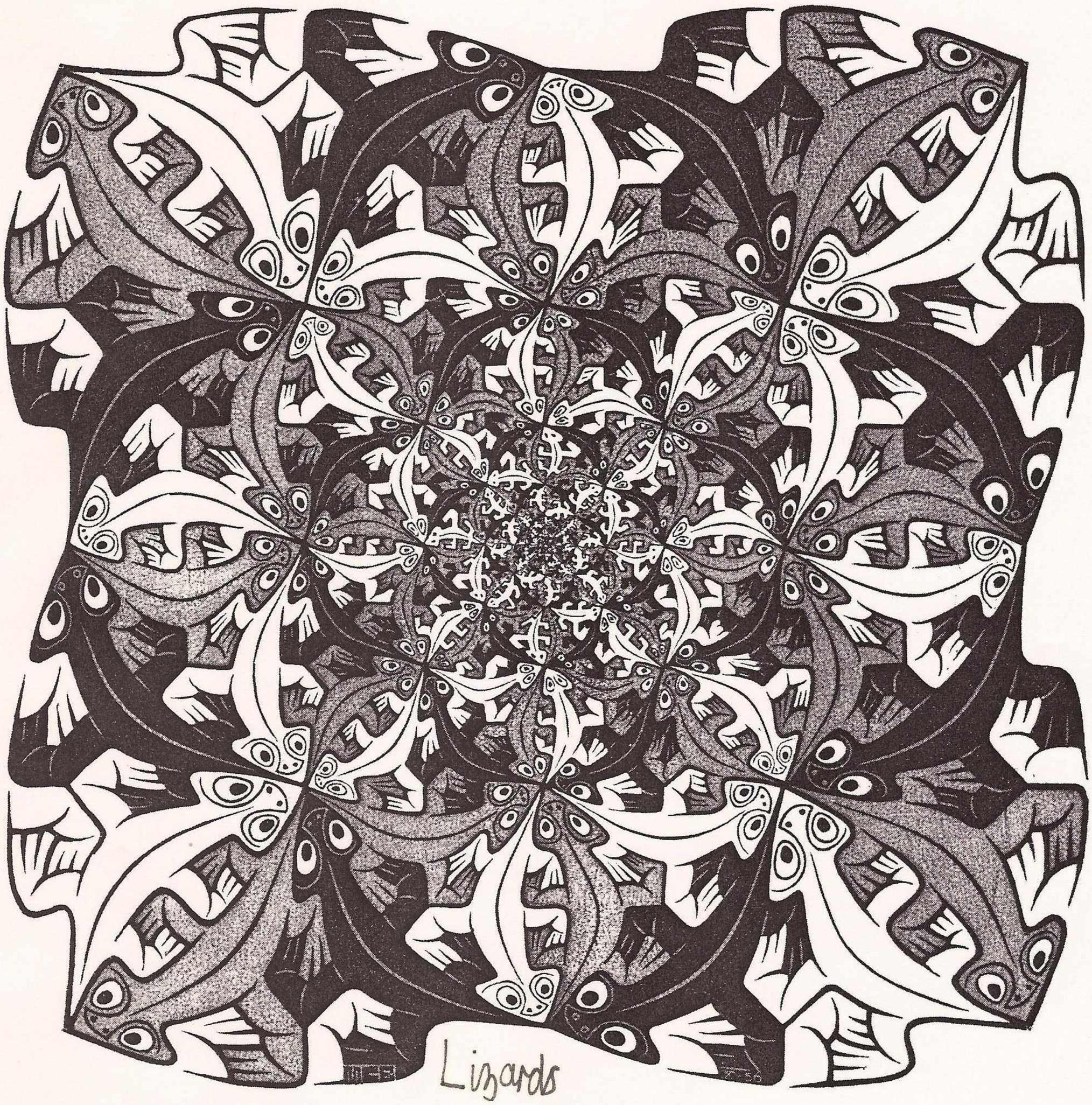
Cirkel-limit IV
Sirkelgrense IV
Cirkelgräns IV
Kehäraja IV
Limite del cerchio IV
Limite circular IV



Fish

22.
Kreislimit I
Circle limit I
Limite circulaire I
Cirkellimiet I
Límite circular I

Cirkel-limit I
Sirkelgrense I
Cirkelgräns I
Kehäraja I
Limite del cerchio I
Limite circular I



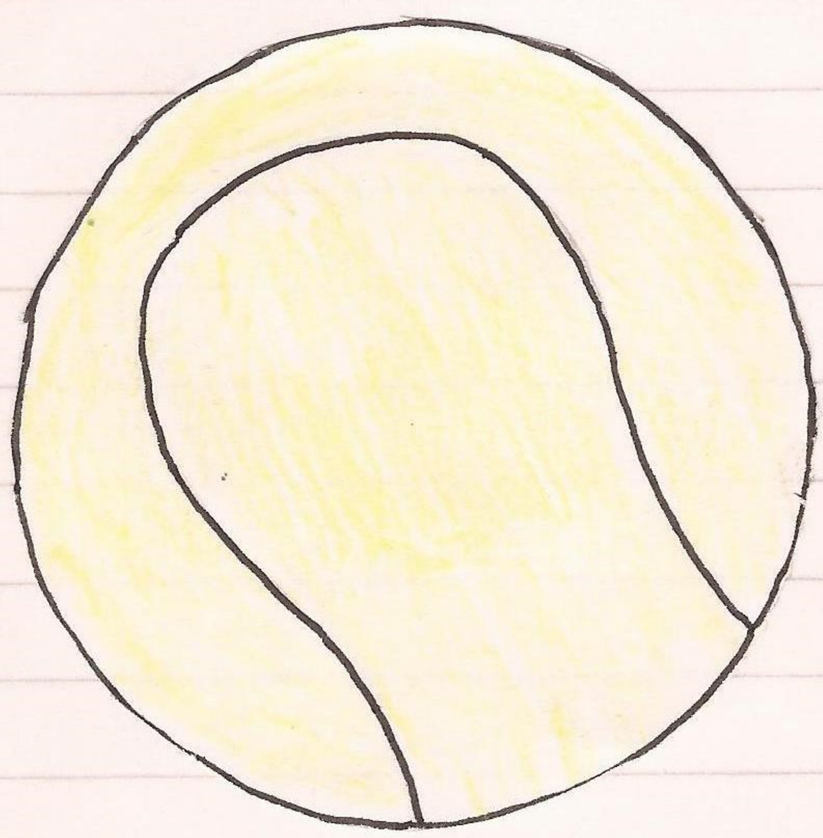
This fractal is the opposite way round; it gets smaller towards the middle.

20.
Kleiner und kleiner
Smaller and smaller
De plus en plus petit
Kleiner en kleiner
Cada vez más pequeño

Mindre og mindre
Mindre og mindre
Mindre och mindre
Pienempi ja pienempi
Sempre più piccolo
Cada vez mais pequeno

FURTHER IDEAS

This project has only covered a very small area of tessellation. It is a very big section of mathematics. Other ideas could be:-

1) 3 dimensional tessellations.
2) Polyhedra tessellation (nets)
3) Tessellation in other dimensions (like hyper-cubes)
4) Tessellation on curved surfaces:-

A tennis ball is a good example of this. Two identical shapes wrap around a sphere interlocking into each other. Here is a picture of one.